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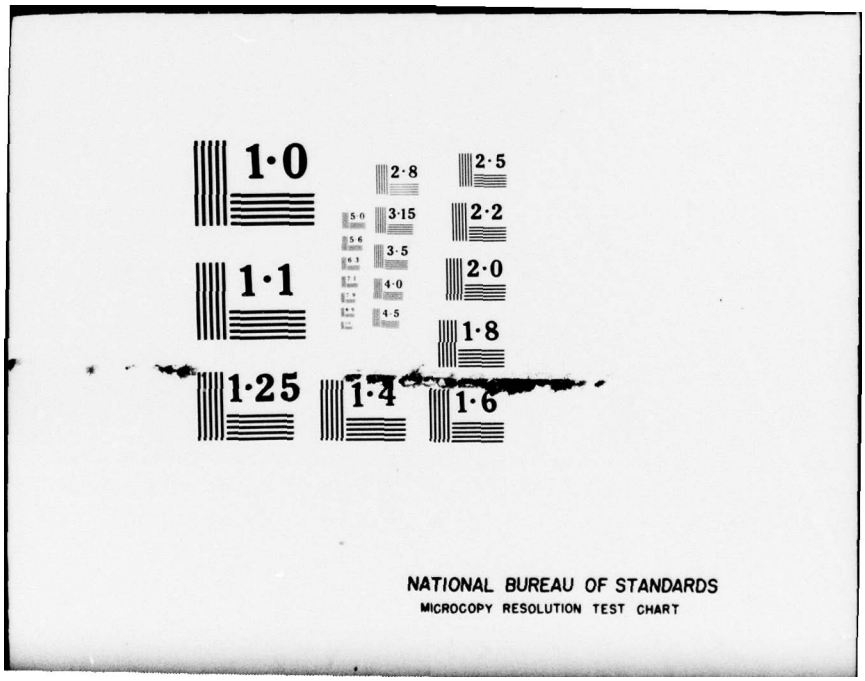
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Sign Changes in Linear Combinations of Derivatives and Convolutions of Pólya Frequency Functions

by

Steven Nahmias[1], University of Pittsburgh

and

Frank Proschan[2], Florida State University

FSU Statistics Report M359R
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The Florida State University
Department of Statistics
Tallahassee, Florida 32306

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

We obtain upper bounds on the number of sign changes of linear combinations of derivatives and convolutions of Pólya frequency functions are obtained using the variation diminishing properties of totally positive functions. These constitute extensions of earlier results of Karlin.

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Sign Changes in Linear Combinations of Derivatives and Convolutions of Pólya Frequency Functions

by

Steven Nahmias[1], University of Pittsburgh

and

Frank Proschan[2], Florida State University

Abstract

We obtain upper bounds on the number of sign changes of linear combinations of derivatives and convolutions of Pólya frequency functions using the variation diminishing properties of totally positive functions. These constitute extensions of earlier results of Karlin.

[1]Research supported by National Science Foundation Grant ENG 75-04990.

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1. Introduction and Summary.

In Karlin and Proschan (1960), results are obtained concerning the number of sign changes of linear combinations of convolutions of sign regular functions, while Karlin (1957 and 1968, pp. 325-326) has obtained upper bounds on the number of sign changes of linear combinations of first and second derivatives of such functions. (For a function f defined on the real line we denote the number of sign changes of f by $S(f) = \sup S[f(t_1), f(t_2), \ldots, f(t_m)]$, where the supremum is extended over all sets $t_1 < t_2 < \ldots < t_m$ on the real line, m is arbitrary but finite, and $S(x_1, \ldots, x_m)$ is the number of sign changes of the indicated sequence, zero terms being discarded. For the definition of sign regularity, see Karlin, 1968, p. 12.) In the present paper, sharper versions of these results are obtained in addition to conclusions regarding linear combinations of both derivatives and convolutions of Pólya frequency functions.

1.1. Definition.

A function f defined on the real line is said to be a Pólya frequency function of order n (PF_n) if $x_1 < \ldots < x_k$, $y_1 < \ldots < y_k$ imply

$$\begin{vmatrix} f(x_1 - y_1) & \ldots & f(x_1 - y_k) \\ \vdots & & \\ f(x_k - y_1) & \ldots & f(x_k - y_k) \end{vmatrix} \geq 0$$

for $k = 1, 2, \ldots, n$.

Note that PF_n functions possess the sign regularity property.

2. Linear Combinations of Derivatives of Pólya Frequency Functions.

The following result is well known (see Karlin, 1968, p. 326); but is included since it is the basis for deriving many of our results. A proof is included for completeness and to illustrate our method of approach.

2.1. Lemma.

Assume that f is a PF_{n+1} function for fixed n = 1, 2, ... Then the n^{th} derivative, $f^{(n)}(x)$, changes sign at most n times. When n sign changes do occur, they occur in the order + - + ··· ±.

Proof. Write

$$f^{(n)}(x) = \lim_{\Delta \downarrow 0} \frac{1}{\Delta^n} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f(x + k\Delta)$$

$$= \lim_{\Delta \downarrow 0} \frac{1}{\Delta^n} \lim_{m \to \infty} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \int_{-\infty}^{\infty} g_m(k\Delta + u)\, f(x - u)\, du,$$

where

$$g_m(u) = \begin{cases} m & \text{for } 0 \le u \le 1/m \\ 0 & \text{elsewhere.} \end{cases}$$

Thus

$$f^{(n)}(x) = \lim_{\Delta \downarrow 0} \frac{1}{\Delta^n} \lim_{m \to \infty} \int_{-\infty}^{\infty} \left[\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} g_m(k\Delta + u) \right] f(x - u)\, du .$$

For fixed $\Delta > 0$ and m sufficiently large, the bracketed sum will have sign pattern + - + ... ± (the final sign being + or - as n is odd or even, respectively). No additional sign changes are introduced as $m \to \infty$ and $\Delta \downarrow 0$. By the variation diminishing property (VDP) of the PF_{n+1} function f (see Karlin, 1968, Chap. 6), the desired result follows. ||

The following result may be established by essentially the same approach as that above and by using the variation diminishing property of PF functions.

2.2. Theorem.

Assume that f is PF_{n+1} for fixed n = 1, 2, ... Then $g(x) = \sum_{j=0}^{n} b_j f^{(j)}(x)$ possesses at most n sign changes.

It is interesting to note that Theorem 2.2 can also be obtained by applying known results. From Theorem 2.1 on p. 50 of Karlin (1968), we find that $\{f, f^{(1)}, f^{(2)}, .., f^{(n)}\}$ comprise a Weak Tchebychev (WT) System. That the generalized polynomial, g(x), possesses no more than n changes of sign follows from Theorem 4.1 on p. 22 of Karlin and Studden (1966) .

However, the essential method of proof for Lemma 2.1 may be used to obtain additional results which are not a consequence of the theory of WT Systems. In particular we have the following.

2.3. Theorem.

Suppose that $P_0, P_1, \ldots, P_n$ are a sequence of nonnegative integers, $a_0, a_1, \ldots, a_n$ are non-zero real numbers, and $t_0 < t_1 < \ldots < t_n$ is a sequence of increasing real numbers. Let f be PF_{w+1}, where

$$w = \sum_{j=0}^{n} P_j + \sum_{j=0}^{n-1} U(a_j, a_{j+1}, P_j) \text{ and}$$

$$U(a_j, a_{j+1}, P_j) = \begin{cases} 1 & \text{if } P_j \text{ is odd and } S(a_j, a_{j+1}) = 0, \text{ or} \\ & \text{if } P_j \text{ is even and } S(a_j, a_{j+1}) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then $g(x) = \sum_{j=0}^{n} a_j f^{(P_j)}(x - t_j)$ possesses at most w sign changes.

Proof.

As in the proof of Lemma 2.1, we may write, after interchanging the integral and the summation sign,

$$g(x) = \lim_{\Delta \downarrow 0} \lim_{m \to \infty} \int_{-\infty}^{\infty} \sum_{j=0}^{n} (a_j/\Delta^{P_j}) \left| \sum_{k=0}^{P_j} \binom{P_j}{k} (-1)^{P_j - k} g_m(k\Delta + u - t_j) \right| \cdot f(x - u)du,$$

where $g_m(\cdot)$ has the same definition as in the proof of Lemma 2.1.

As u approaches $t_j - k\Delta$, the bracketed term will have sign pattern $+ - + \ldots \pm$, the final sign being + if P_j is even and - if P_j is odd. It follows that for u near $t_j - k\Delta$, the term $a_j \cdot [\ ]$ will have sign apttern

$+ - + \ldots +$ if $a_j > 0$ and P_j is even

$+ - + \ldots -$ if $a_j > 0$ and P_j is odd

$- + - \ldots -$ if $a_j < 0$ and P_j is even

$- + - \ldots +$ if $a_j < 0$ and P_j is odd.

Each term $a_j.[\ ]$ will contribute up to P_j sign changes. In addition, there will be one more sign change introduced between the final sign of the string of P_j plusses and minuses at t_j and the first sign of the string of P_{j+1} plusses and minuses at t_{j+1} for $0 \leq j \leq n+1$ if and only if a_j and a_{j+1} have opposite signs for P_j even and like signs for P_j odd.

The number of sign changes is not increased as $\Delta \downarrow 0$ and $m \to \infty$. The result follows by the VDP property of the PF_{w+1} function f. ||

An interesting point to note here is that in general it is not possible to determine w from just the knowledge of $S(a_0, \ldots, a_n)$. However if all

P_j, $0 \le j \le n$, are even numbers (zero included), then it is easy to see that $w = \sum_{j=0}^{n} P_j + S(a_o, \ldots, a_n)$, while if all the P_j are odd numbers, then $w = \sum_{j=0}^{n} P_j + n - S(a_o, \ldots, a_n)$.

3. Linear Combinations of Convolutions of Pólya Frequency Functions.

Denote by $f^{n*}(x)$ the n-fold convolution of a PF function f. We have the following result:

3.1. Theorem.

Let f be a PF_k density with $f(x) = 0$ for $x < 0$. Let $g(x) = \sum_{i=1}^{k} a_i f^{n_i*}(x)$, where $n_1 < n_2 < \ldots < n_k$, and the a_i are real non-zero constants. Then $s(g) \leq S(\underline{a})$. Moreover, if $S(g) = S(\underline{a})$, then the sign changes of $(a_1, \ldots, a_n)$ and of g occur in the same order.

Proof.

The proof is by induction on k. Clearly the result holds for $k = 1$.

Suppose it holds for 1, 2, ..., k - 1. Then write

$$g(x) = \sum_{i=1}^{k} a_i f^{n_i*}(x) = \lim_{m\to\infty} \int_0^\infty [a_1 g_m(u) + \sum_{i=2}^{k} a_i f^{(n_i-n_1)*}(u)] \, f^{n_1*}(x-u)du.$$

where $g_m(u)$ is defined in the proof of Lemma 2.1.

By the inductive hypothesis $S(\sum_{i=2}^{k} a_i f^{(n_i-n_1)*}(u)) \leq S(a_2, \ldots, a_k)$. Since the term $a_1 g_m(u)$ will introduce no additional sign changes in the bracketed term when $S(a_1, a_2) = 0$ and m is sufficiently large, and introduces one additional sign change in the bracketed term if $S(a_1, a_2) = 1$ for m sufficiently large, the result now follows from the variation diminishing property possessed by f.

The proof that if g does possess the full n sign change then they occur in the same order as in $(a_1, \ldots, a_n)$ is simple and so is omitted.

The following result may now be deduced from Theorem 3.1. and the var-

iation diminishing property of TP_k functions.

3.2. Corollary. Let f be PF_k and $f(x) = 0$ for $x < 0$. Then $f^{n*}(x)$ is TP_k in $n = 1, 2, \ldots$ and $x \geq 0$.

Karlin and Proschan (1960) obtain a slightly more general version of Corollary 3.2 by using the original definition of totally positive functions. (See Theorem 1 on p. 724.) Theorem 3.1 above can then be deduced from the fact that $f^{(n)}(x)$ is TP_k in n and x and the variation diminishing property of TP_k functions. We have shown that the same reuslts may be obtained far more simply by proving Theorem 3.1. directly and using the fact that the variation diminishing property characterizes TP_k functions.

When f(x) does not vanish on the negative half line, a proof similar to that of Theorem 3.1. gives the result that g(x) possesses at most $2 \cdot S(\underline{a})$ changes of sign. This is a sharper version of Theorem 8 on p. 730 of Karlin and Proschan (1960).

4. Derivatives and Convolutions.

Although $\{f, f^{(1)}, f^{(2)}, \ldots, f^{(n)}\}$ constitute a WT system, one can show that $\{f, f^{(1)}, \ldots, f^{(n)}, f^{2*}, \ldots, f^{m*}\}$ will not constitute a WT system when $n \geq 1$ and $m \geq 2$. Hence the following is not a consequence of the theory of WT Systems.

4.1. Theorem.

Let $1 < P_0 < P_1 \ldots < P_k$ and $1 < n_1 < \ldots < n_m$ be sequences of non-negative integers and $a_1, \ldots, a_m$ and $b_0, \ldots, b_k$ be sequences of non-zero real numbers. Suppose that f is PF_{w+1} and $f(x) = 0$ for $x < 0$, where

$$w = P_k + S(\underline{a}) + \upsilon(b_k, a_1, P_k). \quad (\upsilon \text{ is as defined in Theorem 2.3.})$$

Then $g(x) = \sum_{i=1}^{m} a_i f^{n_i*}(x) + \sum_{j=0}^{k} b_j f^{(P_j)}(x)$ changes sign at most w times.

Proof.

$$g(x) = \lim_{\Delta \downarrow 0} \lim_{t \to \infty} \int_{-\infty}^{+\infty} \{\sum_{j=0}^{k} b_j \cdot [\sum_{\ell=0}^{P_j} \frac{1}{\Delta^{P_j}} \binom{P_j}{\ell} (-1)^{P_j - \ell} g_t(\ell\Delta + u)]$$

$$+ \sum_{i=1}^{m} a_i f^{(n_i-1)*}(u)\} f(x-u)du$$

$$= \lim_{\Delta \downarrow 0} \lim_{t \to \infty} \int_{-\infty}^{+\infty} \{\sum_{\ell=0}^{P_k} w_\ell(\Delta) g_t(\ell\Delta+u) + \sum_{i=1}^{m} a_i f^{(n_i-1)*}(u)\} f(x-u)\, du,$$

where $w_\ell(\Delta) = \sum_{j=a(\ell)}^{k} (b_j/\Delta^{P_j}) \binom{P_j}{\ell} (-1)^{P_j - \ell}$

and $a(\ell) = i$ if $P_{i-1} < \ell \leq P_i$ for $0 \leq i \leq k$

(interpret $P_{-1} = -1$).

For every fixed value of $\Delta > 0$, the sequence $W_0(\Delta), \ldots, W_{P_k}(\Delta)$ has at most p_k changes of sign (excluding zeroes) starting with $\operatorname{sgn}(b_k)$. This sign pattern will occur arbitrarly close to $u = 0$ by choosing Δ sufficiently small and will always dominate the sign of $\sum_{i=1}^{m} a_i f^{(n_i-1)*}(u)$ at $u = 0$ by choosing t to be sufficiently large.

The term $\sum_{i=1}^{m} a_i f^{(n_i-1)*}(u)$ possesses at most $S(\underline{a})$ sign changes as u traverses R^+ commencing with $\operatorname{sgn}(a_1)$. The final aim of the first group of terms will differ from the first sign of the second group if P_k is even and $S(a_1, b_k) = 1$ or if P_k is odd and $S(a_1, b_k) = 0$.

When f does not vanish on the negative half line, we have the following:

4.2. Theorem.

Assume f is PF_{w+1} and $f(x) \neq 0$ for some $x < 0$. Suppose that $g(x)$ is as given in Theorem 4.1. and $w = p_k + 2\, S(\underline{a}) + c_k$, where $c_k = 1$ if p_k is odd and $= 2$ if p_k is even.

Proof.

As in the proof of Theorem 4.1, the derivative terms in the integrand change sign at most P_k times in a neighborhood around 0. However, in this case the convolution terms may change sign $2\, S(\underline{a})$ times. If P_k is odd then one additional sign change is introduced independent of the sign pattern of $\sum_{i=1}^{m} a_i f^{(n_i-1)*}(u)$. However if P_k is even, two additional sign changes may be introduced if the sign of the convolution terms at zero differs from $\operatorname{sgn}(b_k)$.||

Using essentially the same technique we could determine the maximum number of sign changes of an expression of the form

$$g(x) = \sum_{j=0}^{n} a_j \, f^{(P_j)}(x - t_j) + \sum_{i=1}^{m} b_i \, f^{k_i *}(x - y_i),$$

where $t_o < t_1 < \ldots t_n$

$y_1 < y_2 < \ldots, y_m$.

The exact upper bound will depend upon the relative magnitudes of the t_j's and y_i's and the sign pattern of $(a_o, \ldots, a_n)$ and $(b_1, \ldots, b_m)$.

Define $h(x, n, p) = \frac{d^p}{dx^p} f^{n*}(x)$. Then we have the following result which is similar to Theorem 2.3.

4.3. Theorem.

Suppose that $P_1, \ldots, P_k$ is a sequence of non-negative integers, $a_1, \ldots, a_k$ is a sequence of non-zero real numbers, and $n_1 < n_2 < \ldots < n_k$ is a sequence of increasing positive integers. Let f be PF_{w+1} for

$$w = \sum_{j=1}^{k} P_j + \sum_{j=1}^{k-1} \upsilon(a_j, a_{j+1}, P_j)$$

where $\upsilon(a_j, a_{j+1}, a_i)$ is as defined in Theorem 2.3. Assume that $f(x) = 0$ for $x < 0$. Then $g(x) = \sum_{j=1}^{k} a_j \, h(x, n_j, p_j)$ possesses at most w sign changes commencing with $\operatorname{sgn}(a_1)$.

Proof.

The proof is by induction on k. For $k = 1$ the result follows from Lemma 2.1. and the fact that the variation diminishing property is possessed by the convolution of PF functions.

Suppose now the result holds for 1, 2, ..., k - 1. Then

$$g(x) = \lim_{\Delta \downarrow 0} \lim_{m \to \infty} \int_{-\infty}^{+\infty} \{(a_1 \,/\, \Delta^{P_1}) \cdot \sum_{i=0}^{P_i} \binom{P_i}{i} (-1)^{P_1 - i} g_m(i\Delta + u)$$

$$+ \sum_{j=2}^{k_1} a_j\, h(u, n_j - n_1, P_j)\} \, f^{n_1 *}(x - u)\, du.$$

The term multiplied by a_1 will have P_1 sign changes starting with $\operatorname{sgn}(a_1)$ at $u = -P_1\Delta$, $u = -(P_1 - 1)\Delta$, ..., $u = 0$. The remaining terms will have $\sum_{j=2}^{k} P_j + \sum_{j=2}^{k-1} \sqcup(a_j, a_j\, a_{j+1}, P_j)$ sign changes for $u \geq 0$ commencing with $\operatorname{sgn}(a_2)$ by induction. An additional sign charge is introduced if $S(a_1, a_2) = 0$ and P_1 is odd or if $S(a_1, a_2) = 1$ and P_1 is even.

4.4 Theorem.

Assume that the vectors $\underline{P}$, $\underline{a}$, and $\underline{n}$ are as given in Theorem 4.3 except that $f(x) \neq 0$ for some $x < 0$. Let f be PF_{w+1} for

$$w = \sum_{j=1}^{k} P_j + \sum_{j=1}^{k-1} c_j,$$

where c_j is defined in Theorem 4.2 and $\sum_{j=1}^{o} c_j \equiv 0$. Then $g(x) = \sum_{j=1}^{k} a_j\, h(x, n_j, p_j)$ changes sign at most w times.

Proof.

The result evidently holds for $k = 1$.

Assume it holds for argument 1, 2, ..., k - 1. Then $g(x)$ may be expressed in the same manner as in the proof of Theorem 4.3 except that $h(u, n_j - n_1, P_j)$ does not vanish for $u < 0$. When P_1 is odd, the first set of terms has sign pattern + - ... - for $a_1 > 0$, and - + ... + for $a_1 < 0$ which occur arbitraiiy close to $u = 0$. In either case the second group of terms introduce exactly one additional sign change. Since by induction the second term nas $\sum_{j=2}^{k} P_j + \sum_{j=2}^{k-1} C_j$ and $P_1 + 1$ additional changes are introducted, the result follows. When P_1 is even the first group of terms has sign pattern

+ - ... + or - + ... -. In either case the second group of terms may introduce two additional sign charges if their sign at zero is - or + respectively.||

Note that when f(x) is a probability density that vanishes on the negative half line, $f^{(j)}(x)$ will possess the full j changes of sign. (See Karlin (1968) p. 326.). Hence many of our results hold with equality, and thus we obtain the number of roots of g(x).

In order to illustrate in a simple example how these results might be used, we consider an approximate perishable inventory model developed by Nahmias (1976). For a product with a lifetime of m periods, he obtains

$$w(z) = (1 - \alpha)cz + L(z) + (\theta+\alpha c)\, H(z) - \alpha(\theta + c) \int_0^z H(z - t)\, f(t)dt$$

as the approximate expected cost of ordering to z in each period, where

α = the discount factor

c = ordering cost per unit

L(z) = expected holding and shortage cost when ordering to z

θ = outdate cost

$H(t) = \int_0^t F^{n*}(u)\, du$

f(t) = one period demand density.

When all costs are linear, it follows that

$$w''(z) = (h + r)\, f(z) + (\theta + \alpha c)\, f^{m*}(z) - \alpha(\theta + c)\, f^{(m+1)*}(z),$$

where h and r are the unit holding and shortage costs respectively. From Theorem 3.1, we see that if f is PF_2, then w'' (z) changes sign no more than once and in the order + -. Along with the fact that $W^1(z)$ changes sign once

from - to + (as is demonstrated in Nahmias (1976)). This guarantees that the z^* minimizing $w(z)$ is the zero of $w'(z)$ and that this zero can be found efficiently. As a further description of the behavior of W, Theorem 4.3 shows that $w'''(z)$ changes sign at most four times.

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